The Big Gap

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1 Topic of Research

There are 4 natural numbers a, b, c, d written on a row. The differences $|a-b|$, $|b-c|$, $|c-d|$, $|d-a|$ are written on the next row. The process continues. For example:

7	3	9	2
4	6	7	5
2	1	2	1
1	1	1	1
0	0	0	0

In this example, $a = 7$, $b = 3$, $c = 9$ and $d = 2$

It can be observed that the null row 0, 0, 0, 0 has been obtained.

- Is this a coincidence or can the null row be obtained for any 4 natural numbers?
- What happens if the 4 numbers are real?
- What happens if there are not 4 numbers, but 3, 5, 6 or more?

2 Solutions

2.1 Solution for 4 natural numbers case

Let's replace every number with its remainder of the division by 2 (named modulo 2).
Let's use the same example as in the Topic of Research section (1):

7	3	9	2
4	6	7	5
2	1	2	1
1	1	1	1
0	0	0	0

$\longrightarrow$

1	1	1	0
0	0	1	1
0	1	0	1
1	1	1	1
0	0	0	0

There are $2^4 = 16$ ways to distribute the remainders 1 and 0 on the first row. After maximum 4 steps the null row is obtained, meaning that after maximum 4 steps, all numbers written will be divisible by 2. This is proved in the following table:

Note: The 16 cases are the cases in **bold**

1	**0**	**0**	**0**
OR			
0	**1**	**1**	**1**
1	**0**	**0**	**1**
1	**0**	**1**	**0**
1	**1**	**1**	**1**
0	**0**	**0**	**0**

0	**1**	**0**	**0**
OR			
1	**0**	**1**	**1**
1	**1**	**0**	**0**
0	**1**	**0**	**1**
1	1	1	1
0	0	0	0

0	**0**	**1**	**0**
OR			
1	**1**	**0**	**1**
0	**1**	**1**	**0**
1	0	1	0
1	1	1	1
0	0	0	0

0	**0**	**0**	**1**
OR			
1	**1**	**1**	**0**
0	**0**	**1**	**1**
0	1	0	1
1	1	1	1
0	0	0	0

Having now only numbers that are divisible by 2, we repeat the process replacing each number with its modulo 4. They can be 0 or 2. Similarly, after maximum 4 steps, every number will be divisible by 4.

After maximum $4k$ steps, where k is a natural number, all numbers will be divisible by $2k$. As we can repeat this process indefinitely, after a certain number of steps, the null row will be obtained.

Looking at the table we can also make a series of observations:

- As the subtractions are circular, having n numbers, the n cases in which the numbers are circularly permuted one position are the same.
- The cases in which we replace the 1 with 0 and the 0 with 1 are the same. We will name these two rows "complementary rows".

2.2 Solution for 4 integer numbers case

This case is trivial because after one step, the numbers will become natural.

2.3 Solution for 4 rational numbers case

We can reduce again our demonstration to the natural numbers case. We consider a, b, c and d four rational numbers:

$a = \frac{p_1}{q_1}, \quad (p_1, q_1) = 1$

$b = \frac{p_2}{q_2}, \quad (p_2, q_2) = 1$

$c = \frac{p_3}{q_3}, \quad (p_3, q_3) = 1$

$d = \frac{p_4}{q_4}, \quad (p_4, q_4) = 1$

$\frac{p_1}{q_1}$	$\frac{p_2}{q_2}$	$\frac{p_3}{q_3}$	$\frac{p_4}{q_4}$
$p_1 * q_2 * q_3 * q_4$	$q_1 * p_2 * q_3 * q_4$	$q_1 * q_2 * p_3 * q_4$	$q_1 * q_2 * q_3 * p_4$
...	...	...	...
...	...	...	...
...	...	...	...

We multiply each number by the smallest common multiple of the denominators or simply by the product of denominators, thus reducing the problem to the natural numbers case. Multiplying does not influence the behavior of differences since we will obtain a row of zeros at the end.

2.4 Solution for 4 real numbers case

In this case we can choose a combination of 4 irrational numbers such that we will never achieve the 0-row. Let us consider $a > 1$ and the numbers 1, a, a^2 and a^3 for the first row.
After the first step we obtain the row: $a-1$, $a(a-1)$, $a^2(a-1)$ and a^3-1 which can be written as $(a-1)(a^2+a+1)$.

Let us consider the cubic equation $a^3 = a^2 + a + 1$. If we find the real root of this cubic equation and we replace $a^2 + a + 1$ with a^3 we obtain the initial numbers multiplied by $a - 1$ which does not affect the behavior of the path of steps. Hence, after every step, we will obtain a new row with the same "properties" as the previous one.

Row
1
2
3
...
...
$n+1$

Numbers			
1	a	a^2	a^3
$a-1$	$a(a-1)$	$a^2(a-1)$	$(a-1)(a^2+a+1)$
$(a-1)^2$	$a(a-1)^2$	$a^2(a-1)^2$	$a^3(a-1)^2$
...	...	...	...
...	...	...	...
$(a-1)^n$	$a(a-1)^n$	$a^2(a-1)^n$	$a^3(a-1)^n$

Now, we find the solution of the equation: $a^3 = a^2 + a + 1 \Leftrightarrow a^3 - a^2 - a - 1 = 0$
The discriminant of the cubic equation $ax^3 + bx^2 + cx + d = 0$ has the formula $\Delta = 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2$

The discriminant of our equation is equal to -44, which is negative, so, our equation has one real root and two non-real complex conjugated roots.
Solving it we obtain the following real root: $a \approx 1.8392867552141612 > 1$

So, we cannot obtain the 0-row for every combination of 4 real numbers. Using a similar combination of numbers, we can prove that we cannot obtain the 0- row for 5, 6, $7 \ldots$ numbers.

2.5 Solutions for any column number

Let n be the number of numbers written on the first row. We suppose that at least one written number is different from the others. We work with the remainders modulo 2 of the numbers.

We distinguish 3 different cases:

2.5.1 n is odd

We prove that it is impossible to get the null row for any n numbers randomly chosen, at least one being different from the others.

It is impossible that after a step the number of 1 on a row is odd

If we suppose that the last affirmation is true, then we consider a row having an odd number of ones and the previous row from which it descends

b_{n-1}	b_n	$b_1 = 0$	b_2	b_3	b_4	b_5	b_6	$\ldots$	$b_n - 2$
a_{n-1}	a_n	$a_1 = 1$	a_2	a_3	a_4	a_5	a_6	$\ldots$	$a_n - 2$

Without losing generality, we suppose $a_1 = 1$. A number depends only on 2 numbers of the previous row: the one above and the one above and right.

For example, a_1 depends on b_1 and b_2.

$a_1 = 1$, then $b_1 \neq b_2$. The two cases conduct to complementary rows, so we consider that $b_1 = 0$. Then $b_2 = 1$. If $a_2 = 1$, then $b_3 = 0$. Otherwise, $b_3 = 1$.
Continuing, we observe that for every 1 on the last row, the number above it is different from the above-right number. Here is an example

1	0	1	1	1	0	0	0	0	1	1	1	1	1	1	1
1	1	0	0	1	0	0	0	1	0	0	0	0	0	0	0

Definition: For a row $(a_1, a_2, \ldots, a_n)$ of 1's and 0's, the index $\in 1, 2, 3, \ldots, n$ is called a change if $a_i \neq a_{i+1}$ (we consider $a_{n+1} = a_n$).
So, for an odd number of ones, there will be an odd number of changes on the previous line.

Observation: The number of changes in a row must be even. Proof: Assume $a_1 = 1$. Moving to the right, circularly n places, we come back to a_1. If the number of changes is odd, then a_1 should be 0, which is a contradiction.
So, starting with $b_1 = 0$, an odd number of changes on the line transform b_1 into 0, which is a contradiction.

Before the null row, there will always be a row full of 1. If n is odd, as we proved, it is impossible to get this tow with an odd value of 1s after any step, so the only possibility of this row to exist is to be the starting one. Thus, after just one step, we obtain that all numbers are divisible by 2. Repeating the process, we need that all numbers should have the same remainder modulo 4. In conclusion, we obtain that the numbers must be all equal.

2.5.2 n is even, but not a power of 2

Using backtracking, the row full of 1 must be obtained in order to obtain the null row. This row can be also only obtained from the complementary rows in which 1 and 0 alternate. Then, the number of ones on this row is $n/2$.

If this number is odd, we apply the last result and eventually obtain that the only numbers that conduct to the null row are an alternating sequence of 2 equal numbers.

If it is not, by continuing using backtracking we eventually reach a row with an odd number of ones. For example, when $n = 12$:

0	0	1	0	0	0	1	0	0	0	1	0	3 of 1
0	1	1	0	0	1	1	0	0	1	1	0	6 of 1
1	0	1	0	1	0	1	0	1	0	1	0	6 of 1
1	1	1	1	1	1	1	1	1	1	1	1	12 of 1
0	0	0	0	0	0	0	0	0	0	0	0	null row

For this case, applying the same methods as in the others, we obtain that if $n = 2^k * p$, where k, p are natural numbers and p is odd, then the only configuration of numbers is:

- p equal numbers (equal to A)
- $(2^k - 1) * p$ equal numbers (equal to B)
- each number equal to A is followed by exactly $(2^k - 1)$ numbers equal to B

2.5.3 n is a power of 2

This is the only case in which for every n natural numbers the null row is obtained. To prove this, we created a C++ program that test all the possible combinations of 0 and 1, no matter how they are arranged. By doing this, we proved that after maximum n steps the null row is reached, meaning that all numbers will be divisible by 2.

Then, as we described in the case of 4 numbers, the numbers will eventually become 0.

In order to assure that we have all combinations of 0 and 1 in our program, we generate the numbers between 1 and $n-1$ and we convert them to binary.

Another approach for the same case:

We observe that instead of subtractions we can use addition, the outcomes being the same considering that we work with modulo 2 remainders.

Let $(a_1, a_2, \dots)$ a binary row having the length 2^k. We note $a(r,i)$ the element on the "i" column and at the "r" step. For example: $a(2,1) = a_1 + a_2$; $a(2,2) = a_2 + a_3$; $a(1,i) = ai$; $a(3,1) = a(2,1) + a(2,2) = a_1 + 2 \cdot a_2 + a_3 = a_1 + a_3$.

We prove that we reach the null row for $k = 3$ (a row of length 8), then using the same method, it is easy to prove for any other $k > 3$. We note $ai =$"i" and $ai + aj =$"ij".

Step	Numbers							
1	1	2	3	4	5	6	7	8
2	12	23	34	45	56	67	78	81
3	13	24	35	46	57	68	71	82
4	1234	2345	3456	4567	5678	6781	7812	8123
5	15	26	37	48	51	62	73	84
6	1256	2367	3478	1458	1256	2367	3478	1458
7	1357	2468	1357	2468	1357	2468	1357	2468
8	12345678	12345678	12345678	12345678	12345678	12345678	12345678	12345678

We observe that:

- $a(r, i+1)$ can be obtained from $a(r,i)$ by circularly permutating the indexes one position. For example:
  $a(4,1) = a1 + a2 + a3 + a4$
  $a(4,2) = a2 + a3 + a4 + a5$

- If we note $a(4,i) = b(1,i)$, by repeating the same process we obtain $a(7,1) = b(4,1) = b_1 + b_2 + b_3 + b_4 = a_1 + a_3 + a_5 + a_7$ etc. (see step 7)

  After the eight step we will have all numbers equal to the sum of the all 8 numbers on the first row, which means that we will obtain on the next row the null row.

There are configurations where the null row appears earlier, however, after maximum 8 steps we are sure that we will have the null row (the row where all the numbers are divisible by 2)

Theorem: $a(2^s, 1) = a_1 + a_2 + \ldots +$ for every $s \in 1, 2, ...k$

Particularly, $a(^k, 1) = a(2k, i) = a1+a2+\ldots+$for every $i \in 1, 2, \ldots 2k$, so $a(2^k+1, i) = 0$ for every $i \in 1, 2, \ldots 2^k$

3 Development of the Research Topic

3.1 The Fractal

Let's imagine an empty row of an infinite number of columns with a 1 at the top right and apply the instructions:

...	0	0	0	0	0	0	0	0	0	0	0	1
...	0	0	0	0	0	0	0	0	0	0	1	1
...	0	0	0	0	0	0	0	0	0	1	0	1
...	0	0	0	0	0	0	0	0	1	1	1	1
...	0	0	0	0	0	0	0	1	0	0	0	1
...	0	0	0	0	0	0	1	1	0	0	1	1
...	0	0	0	0	0	0	0	0	0	0	0	1

To make it more visual, let's focus on the upper right part of the table and color the 0 in white and the 1 in black. (1) After an infinite number of steps, we would have a self-similar fractal. This means that the structure and

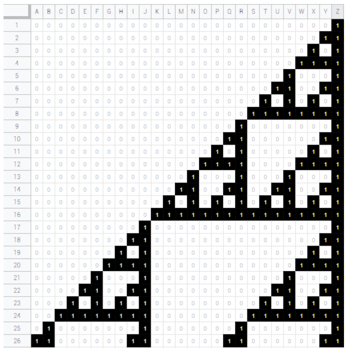

Figure 1: After 26 steps

shape would not change according to the scale. By applying a *2 or *0.5 zoom, we would see exactly the same triangle..., so this tells us that when the triangle repeats itself in the table, it will double its size.

Step	Width
0	1
1	2
2	4
3	8
4	16
...	...
n	2^n

3.2 The Third Line

The goal here was to start from the line of zeros and try to get to the lines before, but not with the modulos of the numbers, with any natural numbers.

3.2.1 Hypothesis

Consider the following:
a, b, c, $d \in \mathbb{N}$ *such that the following table is respected*
$y \in \mathbb{N}_0$

...	...	...	...
a	b	c	d
$\|a-b\| = y$	$\|b-c\| = y$	$\|c-d\| = y$	$\|d-a\| = y$
$y-y=0$	$y-y=0$	$y-y=0$	$y-y=0$

3.2.2 Thesis

We can then prove that the naturals a,b,c,d can only fall in three separate cases:

(Case 1) x-y-x-y

$$a = c$$
$$b = d$$

– – –– **OR** – – ––

(Case 2.1) x-y-x-z

$$a = c$$
$$b,\ d \in \{a \pm y\}$$
$$b \neq d$$

– – –– **OR** – – ––

(Case 2.2) x-y-z-y

$$b = d$$
$$a,\ c \in \{b \pm y\}$$
$$a \neq c$$

3.2.3 Demonstration

$$\begin{cases} |a-b| = |b-c| \\ |b-c| = |c-d| \\ |c-d| = |d-a| \\ |d-a| = |a-b| \end{cases}$$
$$\Leftrightarrow \begin{cases} (a-b)^2 = (b-c)^2 \\ (b-c)^2 = (c-d)^2 \\ (c-d)^2 = (d-a)^2 \\ (d-a)^2 = (a-b)^2 \end{cases}$$
$$\Leftrightarrow \begin{cases} (a-b)^2 - (b-c)^2 = 0 \\ (b-c)^2 - (c-d)^2 = 0 \\ (c-d)^2 - (d-a)^2 = 0 \\ (d-a)^2 - (a-b)^2 = 0 \end{cases}$$
$$\Leftrightarrow \begin{cases} [(a-b) - (b-c)] \cdot [(a-b) + (b-c)] = 0 \quad (A) \\ [(b-c) - (c-d)] \cdot [(b-c) + (c-d)] = 0 \quad (B) \\ [(c-d) - (d-a)] \cdot [(c-d) + (d-a)] = 0 \quad (C) \\ [(d-a) - (a-b)] \cdot [(d-a) + (a-b)] = 0 \quad (D) \end{cases}$$

Let's focus on the first two cases.
If the result of a product is zero, one of the two factors is equal to zero:

(A_1) IF $(a-b)-(b-c)=0$	**(A_2) IF** $(a-b)+(b-c)=0$
$\Leftrightarrow a-b-b+c=0$ $\Leftrightarrow a-2b+c=0$ $\Leftrightarrow a+c=2b$ $\Rightarrow b=\frac{a+c}{2}$	$\Leftrightarrow a-b+b-c=0$ $\Leftrightarrow a-c=0$ $\Rightarrow a=c$

AND

(B_1) **IF** $(b-c)-(c-d)=0$	(B_2) **IF** $(b-c)+(c-d)=0$
$\Leftrightarrow b-c-c+d=0$ $\Leftrightarrow b-2c+d=0$ $\Leftrightarrow b+d=2c$ $\Rightarrow c=\frac{b+d}{2}$	$\Leftrightarrow b-c+c-d=0$ $\Leftrightarrow b-c=0$ $\Rightarrow b=d$

A_1 is not compatible with A_2:
If $b=\frac{a+c}{2}$ and $a=c$, $\Rightarrow b=\frac{a+a}{2} \Leftrightarrow b=a \Rightarrow y=b-a=0$ But y cannot be equal to zero.
We can do the same thing with B_1 and B_2.

A_1 is not compatible with B_1:
If $b=\frac{a+c}{2}$ and $c=\frac{b+d}{2}$, $a \neq b \neq c \neq d$ but it's not possible to have a line like that before the lines of y and zeros (we know that because here, we only use the first two cases but all the cases can be used for all the numbers).

There are three cases left:

- $(A_1$ & $B_2)$ $b=d$ and $d=\frac{a+c}{2}$ **(Thesis Case 2.1)** (The difference between b and a or d and a is $|a-b|=y$)
- $(A_2$ & $B_1)$ $a=c$ and $c=\frac{b+d}{2}$ **(Thesis Case 2.2)** (The difference between a and b or c and b is $|b-c|=y$)
- $(A_2$ & $B_2)$ $a=c$ and $b=d$ **(Thesis Case 1)** (The difference between a and b or c and d is $|a-b|=y$)

4 Conclusion

Here's what we proved:

- If n is odd:
  We will never reach the zero line if the numbers are not equal
- If n is even but not a power of two:
  Some cases are impossible
- If n is a power of two:
  All cases work unless the numbers are real, in which case some combinations do not work
- The Third Line Thesis (3.2.2)

References

The Game of Differences
The Fractal

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