The Roads Of the City

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Contents

Presentation of the problem **2**

Introduction **2**

I First approach: without intersection **3**

1 Introduction **3**

2 Algorithm **3**
2.1 Presentation of the Algorithm 3
2.2 Proof of the Algorithm 4

3 Computer Program **6**

4 To go further **7**
4.1 Program with blocked roads 7

II Second Approach: with Intersections **8**

5 Limitation of the intersection **8**
5.1 Maximal Number of Intersections 8

6 With 3 houses **9**
6.1 3 points forming a triangle with all angles less than or equal to 120° 9
6.1.1 Construction of the Torricelli Point 9
6.1.2 Proof of the shortest road 11
6.2 3 points forming a triangle with an angle greater than 120° 12

7 For more than 3 houses **14**
7.1 4 Points Forming a Rectangle 14
7.2 Generalization with 4 Points 18
7.3 5 points forming a regular pentagon 19
7.4 6 Points Forming a Regular Hexagon 19
7.5 Generalization for $3 \cdot 2^{\alpha}$ houses that form a regular polygon 20
7.6 Concave quadrilaterals 20

III Conclusions **22**

8 Contacts **22**

9 Thanks **22**

Presentation of the Problem

Once upon a time, there was a town with n houses but no road. Unfortunately, it was difficult to go anywhere in this town, especially in rainy weather because the cars had the annoying tendency to get bogged down. After many complains from the inhabitants, the mayor decided to built roads. Therefore, he asked experts to prepare a city plan based on two simple principles:

- Any two houses must be reachable by the road
- It must cost as little as possible, knowing that the cost of the road is linear

What is the optimal construction ?
Is there an algorithm to find such optimal construction ?

Introduction

The goal of this problem is to find the shortest route such that all the houses are linked together.

Two different approaches were developed : the Belgian team focused on the part of the problem without intersection while the Romanian team allowed road intersections. This article will first present the part without intersections then present the part with intersections.

By intersection it is understood an added point (different from the houses) where 2 or more roads meet each others.

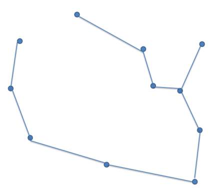


Figure 1: 11 houses linked without intersection

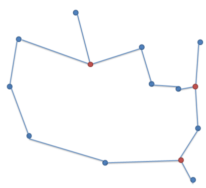


Figure 2: 11 houses linked with 3 intersections

The case without intersection is easier than with intersections. Unfortunately it is less realistic and doesn't give as good results as with intersections.

On the other hand, the case with intersection is more realistic and gives usually better results but is really difficult to prove.

Part I
First approach: without intersection

1 Introduction

Let's first see this problem as a graph problem.

Definition 1.1. a **graph** is a set of vertices (also called nodes) that are linked (or not) by edges

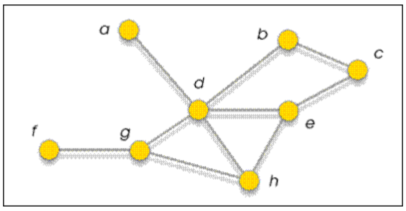


Figure 3: a graph with 8 nodes and 10 edges

Each house will be represented by a nod and each road will be represented by a edge. It also means that every road starts from a house and goes to another. There is no road coming from a house and going to 2 other houses (so no intersections).

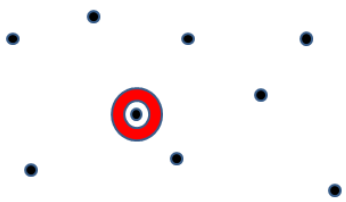
A lot of results about our problem were found . An algorithm to create the smallest route was found and was proven. A program in Python was finally made to find the smallest route on given points. This article will first explain our algorithm, then prove it and finally explain the code developed.

2 Algorithm

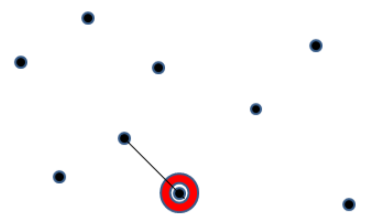
2.1 Presentation of the Algorithm

To find the smallest path for given houses one needs to:

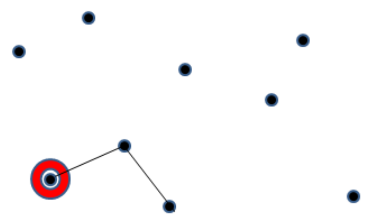
1. take a random house



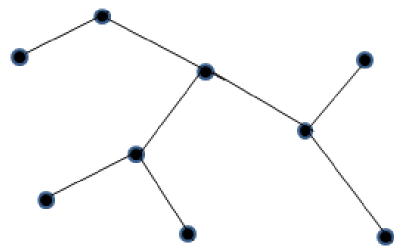
2. link it to the nearest house



3. take the nearest house to one of the houses that is already linked and link it to it



4. repeat the previous step until every house is linked to each other



2.2 Proof of the Algorithm

Before proving the algorithm presented, some vocabulary must be added

Definition 2.1. a **connected** graph is a graph where it is possible to go from the vertex X to the vertex Y for all X, Y

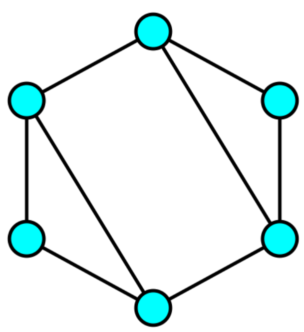


Figure 4: connected graph

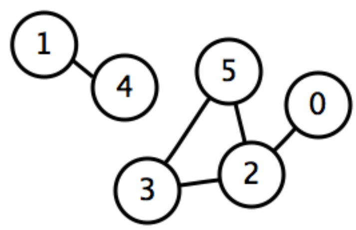


Figure 5: disconnected graph

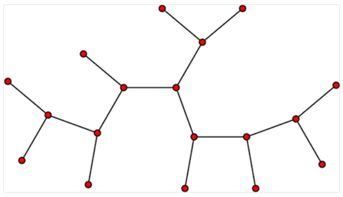


Figure 6: tree

Definition 2.2. a **Tree** is an undirected, acyclic, connected graph. It means that it is a connected graph where there is no cycle and where the segment goes in the 2 directions (There are no one-way "roads")

Lemma 2.3. The smallest path possible is a tree

Proof. The smallest path must be connected, it is in its definition. It must also be acyclic, otherwise one could erase an edge, it will still be connected but strictly smaller. The edges must also be undirected because otherwise there must be at least a cycle to be able to go from each vertices to another, which produce a strictly bigger graph than it would if undirected □

One can now prove that our algorithm produce the smallest tree possible. It will be proven by contradiction. It means that it will be assumed that the algorithm does not produce the smallest path then one will find a contradiction. It will imply that the path produced by the algorithm is (one of) the smallest path.

Proof. :

First, there are m houses. There may exist more than one smallest path (there are for example 3 smallest paths in the case the houses form a equilateral triangle). The smallest paths possible will be called $A_1, A_2, ..., A_n$. As proven in the lemma above, all these A_i are trees.

The different sub-trees produced by the algorithm at each step will also be called $T_0, T_1, T_2, .., T_m$ (T_0 has no edge, T_1 has 1 edge, T_2 is T_1 with one more edge (so 2 edges),...,T_m is a complete tree). To go from T_i to T_{i+1}, one must add the smallest road that link a not-connected house to T_1. It can be sees that T_0 is a sub-tree of every A_i as it is only composed of one house and all the houses are in all the A_i as they are connected graphs.

By the contradiction hypotheses, the algorithm does not lead to a smallest path. However, as T_0 is a sub-tree of every A_i. It means that there is a T_k that is a sub-tree of (without loss of generalities) A_1 and a T_{k+1} that is not the sub-tree of any A_i for every i. ($T_k \in A_1, T_{k+1} \notin A_i, \forall i$)

S is defined as the set of all the nodes that are in T_k

Let denotes by e the edge that is added to T_k to form T_{k+1}. In other words, $T_{k+1} = T_k \cup \{e\}$

The 2 vertices that e links together are denoted by v_1 and v_2. $v_1 \in S, v_2 \notin S$.

As A_1 is connected, there exists a path between v_1 and v_2 in A_1. We call the nodes that are on this path (in order): $v_1 = w_0, w_1, w_2, ...w_{t-1}, w_t = v_2$.

Once again, as $v_1 \in S, v_2 \notin S$, there will be at some point a $w_l \in S$ and $w_{l+1} \notin S$. The edge that goes from w_l to w_{l+1} is called f .

$$A_1 \cup \{e\} \setminus \{f\}$$

is also a smallest path.

Indeed, it can link the same houses than A_1. From w_0 to w_l, it follows the same path than in A_1 and from w_{l+1} to w_t it follows the opposite direction than in A_1: going in the order $w_0, w_t, w_{t-1}, ..., w_{l+1}$. As a consequence this graph is a tree.

Furthermore, by the definition of our algorithm, $|e| \leq |f|$ as e and f are linked to S and the algorithm take the nearest house, taking the smallest edge. As a result, $A_1 \geq A_1 \cup \{e\} \setminus \{f\}$.

Thus, T_{k+1} must be a sub-graph of an A_i. This gives a contradiction. So the tree produced by the algorithm presented is optimal.

□

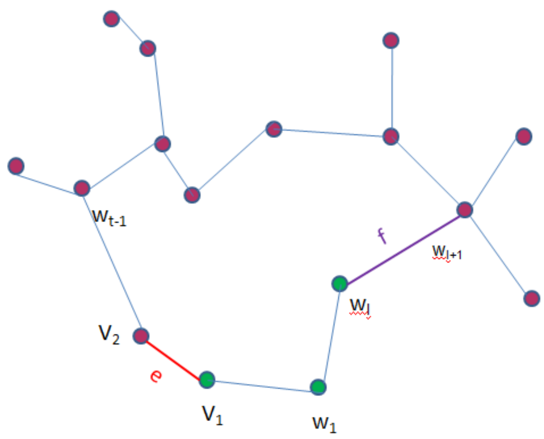


Figure 7: Schema of the proof: In green the vertices in S, in this case, the only difference between A_1 and T_m are the sides f and e. Also, T_3 is a sub-graph of A_1 but T_4 is not a sub-tree of any A_i

3 Computer Program

The Belgian team created a program that put houses (points) at random integer coordinates from 0 to 100 (to simplify the computation). Then it calculates all the different distances. After it follows the algorithm presented and gives the total length of the smallest path. Finally it shows the houses and the routes on a graph.

It also can be easily changed into a program that takes as input a number of houses and their coordinates if one needs a specific disposition

On the next page, the figures 8 and 9 show the result of the program and a schema of the code.

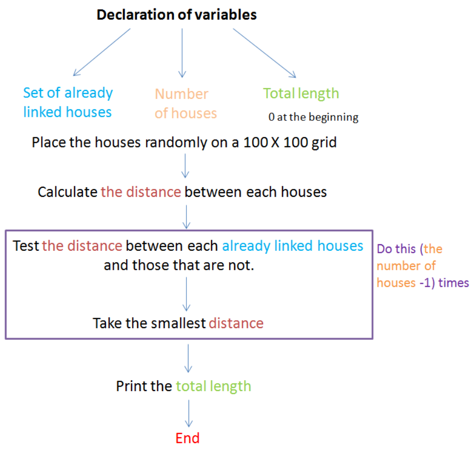


Figure 8: schema of our code

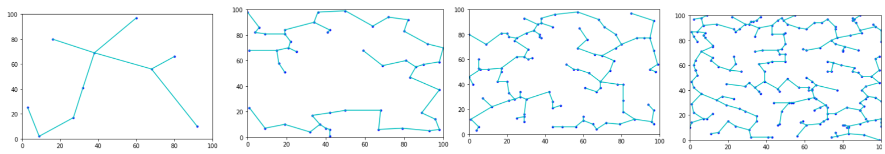


Figure 9: Example with 10,50,100,200 houses

4 To go further

The Belgian team also wondered what would happen if it wasn't possible to link some roads directly together because there was for example a mountain or a lake between them.

In fact, the algorithm that is presented in 2.1 still produce the optimal tree. The only thing that changes is that one only need to check the roads available when choosing the smallest road.

The reader can convince himself that the proof shown above also work with blocked roads.

4.1 Program with blocked roads

A program including blocked roads was also made. It is the same code than the one shown above, with a part that chose randomly some blocked roads and check if the road chosen isn't blocked. The figure 4.1 shows some pictures of the program.

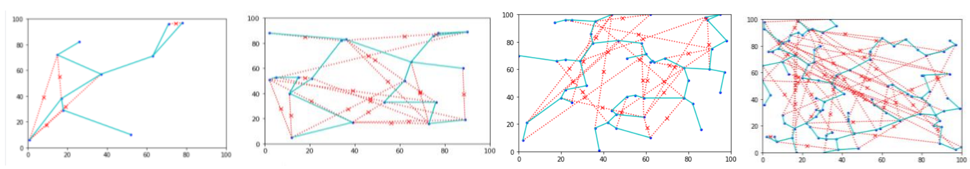


Figure 10: Example with 10/5,20/20,50/25,100/50 houses/blocked roads

Part II
Second Approach: with Intersections

As in the first part, each house is associated to a point. However, this time intersections are allowed.

5 Limitation of the intersection

Lemma 5.1. In order to obtain the shortest route, each intersection point should be connected to at least 3 roads.

Proof. If an intersection point P were connected to only two roads, then one could remove the intersection point and connect directly the two houses. Using the triangle's inequality [1], it is obvious that the second route is shorter. □

5.1 Maximal Number of Intersections

To optimize the route, one have to limit the number of intersection points. It will be done using some elementary elements of graph theory.

Lemma 5.2. Let n be the number of houses, t the number of intersection points and r the number of roads. Then:

$$r = t + n - 1$$

Proof. As mentioned in Lemma 2.3, the shortest route is a tree. The number of segments in a tree is equal to the number of nodes minus 1. In this case the number of nodes is $t + n$ so the conclusion follows. □

Lemma 5.3.

$$r \geq \frac{3t + n}{2}$$

Where n is the number of houses, t the number of intersections and r the number of roads

Proof. Let's count the number of roads: each house is connected to at least one road and each intersection to at least three roads (Lemma 5.1). Summing up, one obtain twice the number of roads, because each road is counted twice: once for each end. Dividing by two, the result is obtained. □

Theorem 5.4.

$$t \leq n - 2$$

Proof. Combining Lemma 5.2 and Lemma 5.3, one has $t + n - 1 \geq \frac{3t+n}{2}$. The above inequality can be obtained with basic algebra. □

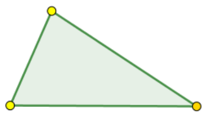
[1] In any triangle ABC, $|AB| \leq |AC| + |BC|$

6 With 3 houses

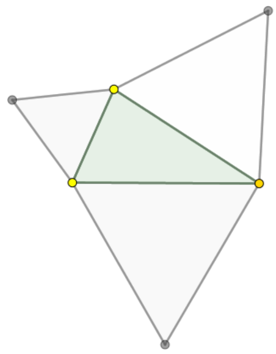
6.1 3 points forming a triangle with all angles less than or equal to $120°$

Let's note the points with A, B, C. In this case, the shortest route passes through the Torricelli point which we denote T.

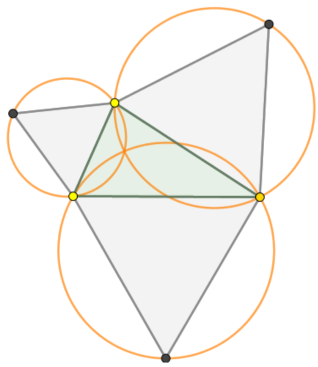
6.1.1 Construction of the Torricelli Point



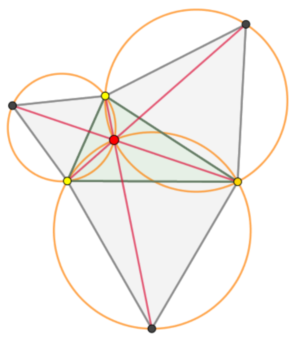
1. One constructs an equilateral triangle on each side of the triangle.



2. One constructs the circumscribed circle of each equilateral triangle.



3. The intersection of the circles is the Torricelli point.



Lemma 6.1. The circumcircle of the equilateral triangles have a common point.

Proof. Let T be the intersection point of circles circumscribed of $\Delta ABC'$ and $\Delta ACB'$.
Then:
Quadrilateral $ATBC'$ is cyclic (it can be inscribed in a circle)
$\Rightarrow |\widehat{ATB}| = 180° - |\widehat{AC'B}| = 180° - 60° = 120°$.
Quadrilateral $ATCB'$ is cyclic
$\Rightarrow |\widehat{ATC}| = 180° - |\widehat{AB'C}| = 180° - 60° = 120°$.
$|\widehat{BTC}| = 360° - |\widehat{ATB}| - |\widehat{ATC}| = 360° - 120° - 120° = 120°$
$\Rightarrow |\widehat{BTC}| + |\widehat{BA'C}| = 180°$
$\Rightarrow$ quadrilateral $BTCA'$ is cyclic
$\Rightarrow T$ is on the circumscribed circle of the $\Delta BCA'$.
$\Rightarrow T$ is the intersection of the circumscribed circles of the equilateral triangles.

□

Lemma 6.2. $|\widehat{ATB}| = |\widehat{ATC}| = |\widehat{BTC}| = 120°$

It can be easily proven using the result and proof of Lemma 6.1

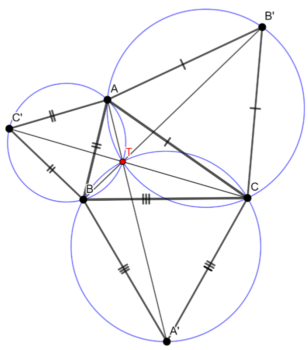
Lemma 6.3. $AA' \cap BB' \cap CC' = \{T\}$

Proof. $BTCA'$ is cyclic $\Rightarrow |\widehat{A'TC}| = |\widehat{A'BC}| = 60°$
$|\widehat{A'TA}| = |\widehat{A'TC}| + |\widehat{CTA}| = 60° + 120° = 180° \Rightarrow T \in AA'$.
Analogous $T \in BB'$ and $T \in CC'$.

□

Lemma 6.4.

$$|AT| + |BT| + |CT| = |AA'| = |BB'| = |CC'|$$



Proof. Applying the Ptolemy's theorem[2] in inscribed quadrilateral $BTCA'$, one obtains:

$$|BC| \cdot |TA'| = |BT| \cdot |A'C| + |A'B| \cdot |CT| \quad (1)$$

$\Delta BCA'$ is equilateral :

$$|BC| = |A'C| = |A'B| \quad (2)$$

From (1) and (2) one obtains: $|TA'| = |BT| + |CT|$.
Using Lemma 6.3, $T \in AA' \Rightarrow |AA'| = |AT| + |TA'| = |AT| + |BT| + |CT|$.
Analogous $|BB'| = |AT| + |BT| + |CT|$ and $|CC'| = |AT| + |BT| + |CT|$.

□

[2]the quadrilateral $ABCD$ is cyclic $\Leftrightarrow |AC| \cdot |BD| = |AB| \cdot |CD| + |AD| \cdot |BC|$

6.1.2 Proof of the shortest road

Let's first introduce some notations: $|BC| = a$, $|AC| = b$, $|AB| = c$, $|AT| = d, |BT| = e, |AC| = f$.

It will first be proven that:

$$|TA| + |TB| + |TC| < |AC| + |AB|$$

Proof. From lemma 6.2, $|\widehat{ATB}| = |\widehat{ATC}| = |\widehat{BTC}| = 120° \Rightarrow cos(T) = -1/2$.

Using the law of cosines in ΔBTC one obtains:

$$|BC|^2 = |BT|^2 + |CT|^2 – 2 \cdot |BT| \cdot |CT| \cdot \cos(T)$$
$$\Leftrightarrow b^2 = e^2 + f^2 + e \cdot f$$

Using the law of cosines in ΔATB one obtains:

$$|AB|^2 = |AT|^2 + |BT|^2 – 2 \cdot |AT| \cdot |BT| \cdot \cos(T)$$

$$\Leftrightarrow c^2 = d^2 + e^2 + d \cdot e$$

So one has:

$$(b+c)^2 = b^2+c^2+2\cdot b\cdot c = e^2+f^2+e\cdot f+d^2+e^2+d\cdot e+2\sqrt{(e^2 + f^2 + e \cdot f)(d^2 + e^2 + d \cdot e)}$$

.

It need to be proven that:

$$b + c > d + e + f$$
$$\Leftrightarrow (b + c)^2 > (d + e + f)^2$$

as b, c, d, e, f are all positives as they are lengths.

$$\Leftrightarrow e^2 + f^2 + e \cdot f + d^2 + e^2 + d \cdot e + 2\sqrt{(e^2 + f^2 + e \cdot f)(d^2 + e^2 + d \cdot e)} > (d + e + f)^2$$

$$\Leftrightarrow e^2+f^2+e\cdot f+d^2+e^2+d\cdot e+2\sqrt{(e^2 + f^2 + e \cdot f)(d^2 + e^2 + d \cdot e)} > d^2+e^2+f^2+2\cdot d\cdot e+2\cdot e\cdot f+2\cdot d\cdot f$$

$$\Leftrightarrow 2\sqrt{(e^2 + f^2 + e \cdot f)(d^2 + e^2 + d \cdot e)} > d \cdot e + e \cdot f + 2 \cdot d \cdot f$$

$$\Leftrightarrow 4(e^2+f^2+e\cdot f)(d^2+e^2+d\cdot e) > d^2\cdot e^2+e^2\cdot f^2+4\cdot d^2\cdot f^2+2\cdot d\cdot e^2\cdot f+4\cdot d\cdot e\cdot f^2+4\cdot d^2\cdot e\cdot f$$

$$\Leftrightarrow 4\cdot e^2\cdot d^2+4\cdot e^4+4\cdot e^3\cdot d+4\cdot f^2\cdot d^2+4\cdot f^2\cdot e^2+4\cdot f^2\cdot d\cdot e+4\cdot e\cdot f\cdot d^2+4\cdot e^3\cdot f+4\cdot d\cdot e^2\cdot f$$

$$>$$

$$d^2 \cdot e^2 + e^2 \cdot f^2 + 4 \cdot d^2 \cdot f^2 + 2 \cdot d \cdot e^2 \cdot f + 4 \cdot d \cdot e \cdot f^2 + 4 \cdot d^2 \cdot e \cdot f$$

$$\Leftrightarrow 3 \cdot e^2 \cdot d^2 + 4 \cdot e^4 + 4 \cdot e^3 \cdot d + 3 \cdot f^2 \cdot e^2 + 4 \cdot e^3 \cdot f + 2 \cdot d \cdot e^2 \cdot f > 0$$

Which is always true

$$\Rightarrow b + c > d + e + f \Leftrightarrow |TA| + |TB| + |TC| < |AC| + |AB|$$

Analogous $|TA| + |TB| + |TC| < |BC| + |AC|$ and $|TA| + |TB| + |TC| < |BC| + |AB|$. □

Now it will be proven that it is the only point where it is obtain the minimal length of the road.

Proof. Let M be a point inside ΔABC. Applying the Ptolemy's inequality in the quadrant $MBA'C$, one obtains : $|MA'| \cdot |BC| \leq |MB| \cdot |A'C| + |MC| \cdot |A'B|$.

$\Delta BCA'$ is equilateral $\Rightarrow |BC| = |A'C| = |A'B|$.

Thus the following equation can be obtained:

$$|MA'| \leq |MB| + |MC| \tag{3}$$

Using the triangle inequality in $\Delta AMA'$, one obtains:

$$|AA|' \leq |MA| + |MA'| \tag{4}$$

From (3) and (4) one obtains :

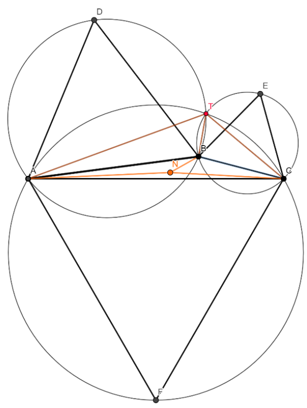
$$|AA'| \leq |MA| + |MB| + |MC|$$

Moreover, one of the properties Torricelli point is : $|AA'| = |TA| + |TB| + |TC|$ (Lemma 6.4). It implies that : $|TA| + |TB| + |TC| \leq |MA| + |MB| + |MC|$. The equality occurs only if $M = T$. □

6.2 3 points forming a triangle with an angle greater than 120°

Let's denote the points with A, B, C and without limiting generality it is assumed that $|\widehat{ABC}| > 120°$.

In this case, the shortest way is obtained without intersections and is represented by $|AB| + |BC| \Rightarrow |AT| + |BT| + |CT| > |AB| + |BC|$.



Proof.

$$|\widehat{ABT}| + |\widehat{TBC}| = 360° - |\widehat{ABC}|$$

But $|\widehat{ABC}| > 120° \Rightarrow -|\widehat{ABC}| < -120°$ Then

$$|\widehat{ABT}| + |\widehat{TBC}| < 240° \tag{5}$$

There are 2 cases:

a) $|\widehat{ABT}| \geq 90°$ and $|\widehat{CBT}| \geq 90°$
(but not $|\widehat{ABT}| = 90°$ and $|\widehat{CBT}| = 90°$)
Using the law of cosines in ΔABT, one obtains:
$|AT|^2 = |BT|^2 + |AB|^2 – 2 \cdot |BT| \cdot |AB| \cdot \cos|\widehat{ABT}|$
But

$$|\widehat{ABT}| \geq 90° \Rightarrow \cos|\widehat{ABT}| \leq 0$$

$$\Rightarrow –2 \cdot |BT| \cdot |AB| \cdot cos|\widehat{ABT}| \geq 0$$

$$\Rightarrow |AT|^2 \geq |BT|^2 + |AB|^2$$

$$\Rightarrow |AT| > |AB| \tag{6}$$

Using the law of cosines in ΔCBT one obtains:
$|CT|^2 = |BT|^2 + |BC|^2 – 2 \cdot |BT| \cdot |BC| \cdot \cos|\widehat{CBT}|$
But

$$|\widehat{CBT}| \geq 90°$$

$$\Rightarrow \cos|\widehat{CBT}| \leq 0$$

$$\Rightarrow –2 \cdot |BT| \cdot |BC| \cdot \cos|\widehat{CBT}| \geq 0$$

$$\Rightarrow |CT|^2 \geq |BT|^2 + |BC|^2$$

$$|CT| > |BC| \tag{7}$$

From (6) and (7) one can obtain:

$$|AT| + |CT| > |AB| + |BC|$$

$$\Rightarrow |AT| + |BT| + |CT| > |AB| + |BC|$$

b) $|\widehat{ABT}| < 90°$ or $|\widehat{CBT}| < 90°$
Using (5) one obtains that $|\widehat{CBT}| > 90°$ or $|\widehat{ABT}| > 90°$
If $|\widehat{CBT}| < 90°$ the triangle inequality can be used for ΔBCT and it implies that:

$$|BT| + |CT| > |BC| \tag{8}$$

From (6) and (8) one obtains:

$$|AT| + |BT| + |CT| > |AB| + |BC|$$

.
If $|\widehat{ABT}| < 90°$ one can use the triangle inequality for ΔBAT and obtain:

$$|BT| + |AT| > |AB| \tag{9}$$

From (7) and (9) we obtain:

$$|AT| + |BT| + |CT| > |AB| + |BC|$$

.
Analogous it can be demonstrated that $|AM| + |BM| + |CM| > |AB| + |BC|$, for any $M \in\ ext\ \Delta ABC$. Analogous one demonstrates $|AN| + |BN| + |CN| > |AB| + |BC|$, for any $N \in\ int\ \Delta ABC$. □

7 For more than 3 houses

Analyzing the case with 3 houses, one obtains some rules that the shortest route has to follow.

Definition 7.1. A Torricelli-like point is an intersection with 3 angles equal to 120° around it.

Properties:

1. Every two roads that meet should form an angle grater or equal than 120°
2. No point should be connected to more than three roads;
3. Any point connected with three roads has to be a Torricelli-like point.
4. All intersection points has to be a Torricelli-like point.

Proof. If two roads connecting in three points, A, B, C form an angle smaller than 120 degrees at their common point node B, then we could construct another intersection point O such that $|OA| + |OB| + |OC|$ is smaller than $|AB| + |BC|$.

As the full circle is only 360 degrees, if a node N were to be connected to 4 or more roads, there would be at least 2 roads connecting N that would form an angle smaller than 120 degrees. For the same reason, if a node is connected by three roads, no two roads connecting it can form an angle larger than 120 degrees : if that were the case, the remaining road would from an angle smaller than 120 degrees with at least one of the two roads.

An intersection point is connected to at least three roads by Lemma 5.1. We just proved an intersection point could not be connected to more than three roads therefore, an intersection point is always connected to exactly 3 roads.

□

7.1 4 Points Forming a Rectangle

Being a particular case, the rectangles offer more than one way to prove our algorithm. The demonstration of the algorithm for rectangles using only geometrical methods will be presented in this part. Let's note the points with A, B, C and D. To simplify the calculations, the following notations will also be used: $|AB| = L, |BC| = l, |GE| = x$

The points E and F will also be introduced such that:

E and F are points in the interior of the rectangle $ABCD$ and $|\widehat{AED}| = 120° = |\widehat{BFC}|$

$|AE| = |DE|$ and $|BF| = |FC|$

With those properties, it can be implied that :

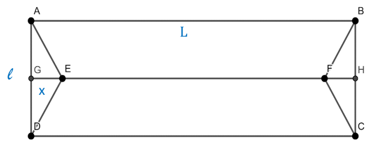
ΔAED is isosceles $\Rightarrow \Delta AGE \equiv \Delta GED \Rightarrow |\widehat{AEG}| = |\widehat{DEG}| = \frac{|\widehat{AED}|}{2} = \frac{120°}{2} = 60°$.

$|\widehat{AEF}| = 180° - |\widehat{AEG}| = 180° - 60° = 120°$

$|\widehat{DEF}| = 180° - |\widehat{DEG}| = 180° - 60° = 120°$

Thus one obtains that $|\widehat{AED}| = |\widehat{AEF}| = |\widehat{DEF}| = 120° \Rightarrow E$ is the Torricelli point for ΔADF.

Similarly, F is the Torricelli point of ΔADF



It will first be proven that: $|AE|+|DE|+|EF|+|BF|+|CF| < L+2\cdot l$ (one compares with the shortest route without intersections)

Proof. $|\widehat{AED}| = 120°$

$|AE| = |DE| \Rightarrow \Delta ADE$ is isosceles

$\Rightarrow |\widehat{DAE}| = |\widehat{ADE}| = 30°$

In $\Delta AGE : |\widehat{GAE}| = 30°,\ |\widehat{GAE}| = 90° \Rightarrow |AE| = 2\cdot|GE| = 2\cdot x$

$$\Rightarrow |AG| = |GE|\sqrt{3} = x\sqrt{3} \tag{10}$$

But G is the midpoint of $[AB]$

$$\Rightarrow |AG| = |GD| = \frac{|AD|}{2} = \frac{l}{2} \tag{11}$$

From 10 and 11 one obtains:

$$x\sqrt{3} = \frac{l}{2}$$
$$\Rightarrow x = \frac{l}{2\sqrt{3}}$$

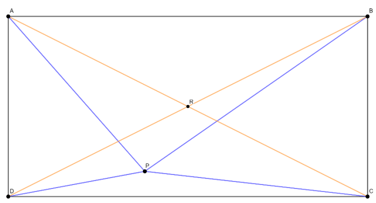
$|EF| = L - 2\cdot|GE| = L - 2\cdot x.$

It must be proven that:

$$L + 2\cdot l > |AE| + |DE| + |EF| + |BF| + |CF|$$
$$\Leftrightarrow L + 2\cdot l > 4\cdot 2\cdot x + L - 2\cdot x$$
$$\Leftrightarrow L + 2\cdot l > L + 3\cdot 2\cdot x$$
$$\Leftrightarrow 2\cdot l > 3\cdot 2\cdot x$$
$$\Leftrightarrow 2\cdot l > 3\cdot 2\cdot \frac{l}{2\sqrt{3}}$$
$$\Leftrightarrow 2\cdot l > \sqrt{3}\cdot l$$
$$\Leftrightarrow 2 > \sqrt{3}$$

Which is always true □

It will be proven that it is shorter than the shortest route with a single intersection



Proof. For starters, let's prove that the shortest route with a single intersection passes through the intersection of the diagonals.

Let P be a point in the interior of the rectangle $ABCD$ and R be the intersection of the diagonals.

From the triangle inequality in ΔACP one obtains that $|AC| < |AP| + |PC|$.

From the triangle inequality in ΔBDP one obtains that $|BD| < |DP| + |PB|$.

From these two inequalities one gets that:

$|AR| + |BR| + |CR| + |DR| = |AC| + |BD| < |AP| + |BC| + |CP| + |DP|$

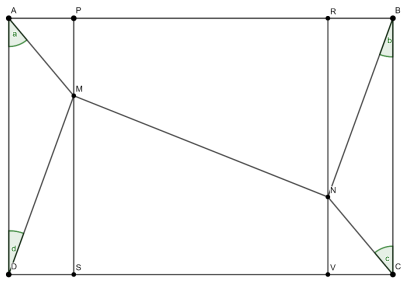
$\Rightarrow$ The shortest route with a single intersection passes through the intersection of diagonals.

Now let's compare the shortest road with a single intersection with the shortest road with 2 intersections.

$$\begin{gathered} |AR| + |BR| + |CR| + |DR| > |AE| + |DE| + |EF| + |BF| + |CF| \\ \Leftrightarrow 2 \cdot \sqrt{L^2 + l^2} > 4 \cdot 2 \cdot \frac{l}{2\sqrt{3}} + L - 2 \cdot \frac{l}{2\sqrt{3}} + \sqrt{3} \cdot L \\ \Leftrightarrow 12 \cdot (L^2 + l^2) > 9 \cdot l2 + 2 \cdot 3 \cdot \sqrt{3} \cdot l \cdot L + 3 \cdot L^2 \\ \Leftrightarrow (3 \cdot L)^2 - 2 \cdot 3 \cdot \sqrt{3} \cdot l \cdot L + (\sqrt{3} \cdot l)^2 > 0 \\ \Leftrightarrow (3 \cdot L - \sqrt{3} \cdot l)^2 > 0 \end{gathered}$$

Which is always true □

It will now be proven that it is the smallest route with 2 intersections



Proof. Let M and N be 2 points in the interior of rectangle $ABCD$. Let T_1 and T_2 be the Torricelli points for respectively ΔADN and ΔBCM.
T_1 - Torricelli point for ΔADN:

$$|AT_1| + |DT_1| + |NT_1| \leq |AM| + |DM| + |NM| \tag{12}$$

T_2 - Torricelli point for ΔBCM:

$$|BT_2| + |CT_2| + |MT_2| \leq |BN| + |CN| + |MN| \tag{13}$$

From (12) and (13) one obtains that M is the Torricelli point for ΔADN, respectively N is the Torricelli point for ΔBCM.

Notations: $|\widehat{DAM}| = a, |\widehat{NBC}| = b, |\widehat{BCN}| = c, |\widehat{ADM}| = d$.

$$\Delta ADM : a + d + 120^\circ = 180^\circ \Rightarrow d = 60^\circ - a$$

$$b = 90^\circ - (360^\circ - 120^\circ - 120^\circ - (90^\circ - a)) = 60^\circ - a$$

Then $b = d$.

$$\Delta ADM : a + d + 120^\circ = 180^\circ \Rightarrow a = 60^\circ - d$$
$$c = 90^\circ - (360^\circ - 120^\circ - 120^\circ - (90^\circ - d)) = 60^\circ - d$$

Then $a = c$.

ΔADM and $\Delta CBN : a = c, d = b, |AD| = |BC| (= l) \Rightarrow \Delta ADM \equiv \Delta CBN$

$$\Rightarrow |DM| = |BN| \text{ and } |AM| = |CN|$$

Let $R \in [AB]$ be a point so that $NR \perp AB \Rightarrow |\widehat{NRB}| = 90^\circ$

$$|\widehat{NBR}| = 90^\circ - b = 90^\circ - d$$

Let $S \in [CD]$ be a point so that $MS \perp CD \Rightarrow |\widehat{MSD}| = 90^\circ$

$$|\widehat{MDS}| = 90^\circ - d$$

ΔSDM and ΔRBN : $|\widehat{NRB}| = |\widehat{MSD}| (= 90^\circ),\ |\widehat{NBR}| = |\widehat{MDS}| (= 90^\circ - d),$

$$|DM| = |BN| \Rightarrow \Delta SDM \equiv \Delta RBN$$

$$\Rightarrow |SM| = |RN|$$

Let $P \in [AB]$ be a point so that $MP \perp AB$ and $V \in [CD]$ be a point so that $NV \perp CD$

Analogous one demonstrates that $|MP| = |NV|$.

Also one obtains that $|\widehat{MSD}| = |\widehat{MPA}| = |\widehat{PAD}| = |\widehat{SDA}| = 90^\circ \Rightarrow APSD$ is a rectangle

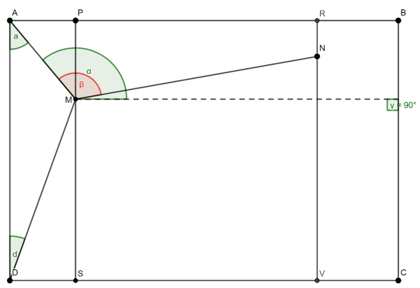
$$|DS| = |AP| \tag{14}$$

Similarly one obtains that $|VC| = |RB|$.

Let assume that $a > d \Rightarrow |MD| > |MA|$, it will be shown by contradiction that it is not possible.

It can be obtained from Pythagoras' theorem in ΔMDS and ΔMPA that $|MS| = \sqrt{|MD|^2 - |DS|^2}$ and $|MP| = \sqrt{|MA|^2 - |AP|^2} \Longrightarrow |MS| > |MP|$

$$\Longrightarrow |NR| > |NV| \tag{15}$$



In the figure there is:

$\alpha = 180^\circ - (90^\circ - a) = 90^\circ + a$

$\beta = 120^\circ$

Because $a + d = 60°$ and $a > d \Rightarrow a > 30° \Rightarrow \alpha > 90° + 30° = \beta \Rightarrow$ The line MN is oblique so that the distance from N to AB is less than the distance from M of $|AB| \Rightarrow |NR| < |MP|$

$$\Longrightarrow |NR| < |NV| \tag{16}$$

From (15) and (16) there is a contradiction.⇒ The assumption made is false. $\Rightarrow a \leq d$.

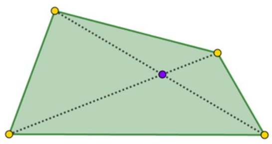
Analogous one obtains a contradiction for $a < d \Rightarrow a = d = \frac{60°}{2} = 30° \Rightarrow \Delta MDA$ is isosceles $\Rightarrow |AM| = |DM|$.

$a = d = 30° \Rightarrow b = c = 30° \Rightarrow \Delta NBC$ is isosceles $\Rightarrow |BN| = |CN|$. Because M and N have all the properties of the points E, respectively F it implies that $M = E$ and $N = F$. □

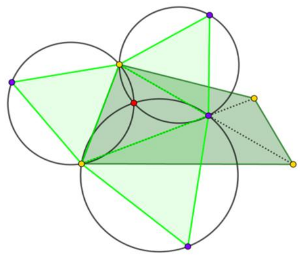
7.2 Generalization with 4 Points

Based on Theorem 5.4 the Romanian team observed that an general optimization of the route consists in having 2 intersection points. However, these two points can vary depending on the quadrilateral's sides.

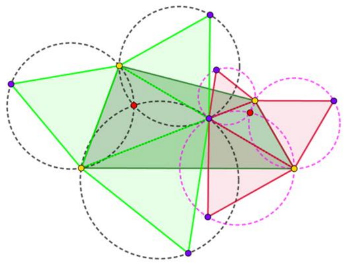
For example, one can create an optimization of the route by taking the quadrilateral diagonals



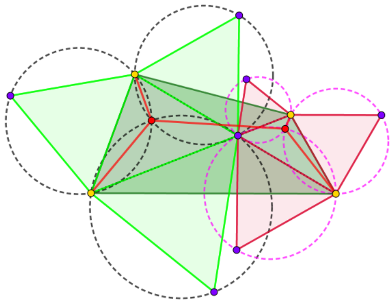
Then one takes the Torricelli point of one of the created triangles (2) quadrilateral vertices and the center of it)



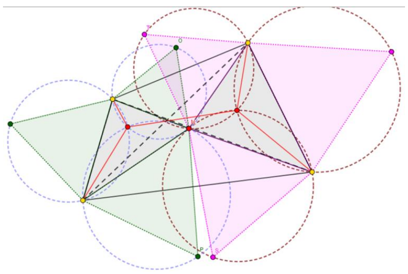
Then do the same with the triangle formed by the other two vertices and the center.



And in the end one unites the two Torricelli points and we have an optimum route for this quadrilateral. (colored in red)

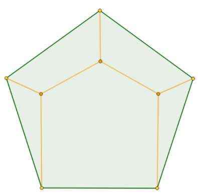


But for other quadrilaterals a better optimization is the one that uses the same algorithm but instead of the intersection of the diagonals, uses their center of gravity of the quadrilateral.



7.3 5 points forming a regular pentagon

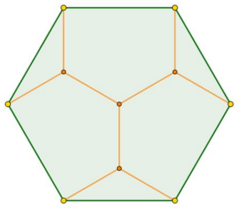
We look for an optimum route with the maximum intersection points : three. From Lemma 5.1 (In order to obtain the shortest route, each intersection point should be connected to at least 3 roads.) and Lemma 5.2 (Let n be the number of houses, t the number of intersection points and r the number of roads. Then: $r = t + n - 1$ in this case $r = 7$;), it results that no two cities/points can be connected directly and that no intersection point can be connected to three cities as the angle between 2 cities is smaller than 120°. So there must be two intersections which connect two cities each. They are connected between them and with the last city by a third intersection, as in the figure below.



We denote by R the length between the center of the pentagon and any of its vertices. This road is $4.57R$ long, the road without intersection is $4.7R$ long

7.4 6 Points Forming a Regular Hexagon

Intuitive we connected the 6 points in 3 pairs of two adjacent cities. Then we connect those 3 resulted intersections into the center of the hexagon. This way we obtain the maximum number of intersection points: four.



Again, let's denote by R the length between the center of the pentagon and any of its vertices. The road length $3\sqrt{3}R = 5.196R$. However, this is longer that the road with no intersection points which is 5R

7.5 Generalization for $3 \cdot 2^{\alpha}$ houses that form a regular polygon

As mentioned above, it is difficult to control more than 3 points, but in this certain case, there is an algorithm that works for any value of α.

Using the properties i, ii, iii, iv and Theorem 5.4 one gets that the shortest route with intersections would be constructed as follows:

one make a partition of the houses in pairs of neighbouring houses. By neighbouring houses it is understood 2 houses A_i and A_{i+1} that lie at the end of the same side of the regular polygon. Each pair of houses A_i, A_{i+1} is linked to an intersection point S such as $|SA_i| = |SA_{i+1}|$ and $|\widehat{A_i S A_{i+1}}| = 120°$. The intersection points obtained form a regular polygon with $3 \cdot 2^{\alpha-1}$ vertices, which is treated as previous.

By induction, this allows to compute the length of the route.

Let R be the radius of the circumscribed circle of the polygon. Then, the distance from the center of the circle to the polygon's vertices is equal to $R \cdot \cos(\frac{180°}{n}) - R \cdot \frac{\sin(\frac{180°}{n})}{\tan(60°)}$. The distance between one intersection point and one house that is linked to it is $R \cdot \frac{\sin(\frac{180°}{n})}{\sin(60°)}$. Therefore, if l_α is the length of the road with $n = 3 \cdot 2^{\alpha}$ houses, then:

$$l_{\alpha+1} = R\frac{\sin(\frac{180°}{n})}{\sin(60°)} + \left(R \cdot \cos\left(\frac{180°}{n}\right) - R\frac{\sin(\frac{180°}{n})}{\tan(60°)}\right) \cdot l_\alpha$$

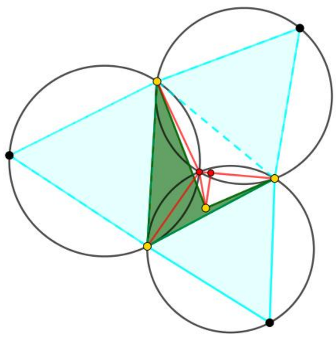
Making some calculus, it can be discovered discover that for $\alpha > 0$, the road without intersection is shorter than the one with intersections.

In the table bellow it is shown the length of the road with intersections for various values of α ($R = 1$).

Number of Vertices	Length of the Road with Intersections
3	3
6	5.19615
12	7.82894
24	10.7892
48	13.9837
96	17.339

7.6 Concave quadrilaterals

For concave figures, it is even harder to develop an algorithm that works for a wide range of cases. In this part an optimization for the case of four houses that form a concave quadrilaterals will be presented, as it is shorter than the one without intersections. Firstly, one can split the quadrilateral in 2 parts: one made from the outside layer of houses, which forms a triangle, and one from the inside point. one builds the Torricelli's point for the triangle, as well as the roads that connect it with the 3 houses. Afterwards, from the left house, one builds a perpendicular road to the nearest already built road, as seen in the figure bellow. However, for a slightly different figure, the smallest path could be really different.



Part III
Conclusions

The case without intersection is easier to prove and give quite good results. However, it usually doesn't give the best result.

In the other hand, the best results are usually found in the case with intersection and are nearer to the reality . However it is really difficult to find an algorithm for a general case. Indeed, this problem is a NP(-hard) problem. It means that there is no algorithm resolving this problem in polynomial time (in time that is given by a polynomial depending of the number of houses). It implies that for large number of houses, there is no algorithm found nowadays. Finding an algorithm for this type of problem (or proving there is not) is a problem of the millennial from the Clay institute.

This research subject enable us to discover the graph theory. It also allowed us to discover the work of the researchers. Our school twinning was an opportunity to discover each other's culture and also the way Mathematics is practised. It was an great way to enhance international cooperation

8 Contacts

If you have questions, remarks or if you would like to have the program we made, please send a mail at simonmartin462@gmail.com, we will be pleased to answer you.

9 Thanks

We would like to thank all our teachers, Mr Sébastien KIRSCH, Mrs Anne LACROIX and Mr Daniel NGUYEN from Collège Sainte-Véronique, Liège and Mrs Gabriela Elena ZANOSCHI from the National College, Iași that helped us and spend time for us throughout this year. We would also like to thank our researcher Mr Julien LEROY from l'Université de Liège and Dr. Eugen VĂRVĂRUCĂ from the University Alexandru Ioan Cuza, Iași that suggested the problem and guided us during our researches . We also thank the people that corrected and checked this article.

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